

ACCURATE THERMAL STRESSES FOR BEAMS: NORMAL STRESS

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Formulations for a general theory of thermoelasticity to generate accurate thermal stresses for structural members of aeronautical vehicles were developed in 1954 by Boley. The formulation also provides three normal stresses and a shear stress along the entire length of the beam. The Poisson effect of the lateral and transverse normal stresses on a thermally loaded beam is taken into account in this theory by employing an Airy stress function. The Airy stress function enables the reduction of the three-dimensional thermal stress problem to a two-dimensional one. Numerical results from the general theory of thermoelasticity are compared to those obtained from strength of materials. It is concluded that the theory of thermoelasticity for prismatic beams proposed in this paper can be used instead of strength of materials when precise stress results are desired.

Keywords: *Thermal Stresses, Elasticity, Normal Stress, and Airy Stress Function*

1. Introduction

Thermal stresses have traditionally been analyzed by using strength of materials [1]. It is the contention here that this approach may produce an error that could be significant to a design engineer. Engineers not only need tools to determine accurate thermal stresses, but also need to know when to use these tools [2]. The difficulty in evaluating the Airy stress function, and the relatively small effect of the terms containing the function on the final thermal stress in thin aeronautical structural members, resulted in the Airy stress function being ignored in the calculations. With the aid of finite elements and computational tools, the previously neglected Airy Stress function in Boley's elasticity theory of thermal stresses can be implemented [1]. The objectives of this paper are to present a new general theory of thermoelasticity for prismatic beams, present results from applying the theory, and determine a set of criteria for the use of the theory by interpreting the results.

2. Thermoelasticity

The elasticity theory of thermal stresses presented in this paper originates from Boley's *Theory of Thermal Stresses* [3]. The equation for the normal stress induced by a temperature change is derived by using general thermoelastic equations, the force method, and a semi-inverse method.

The ends of the beam and lateral surfaces are considered to be traction free. The temperature along

the span (x -direction) varies linearly and can be arbitrary over the cross-section (y and z -directions); i.e.,

$$T(x, y, z) = T^1(y, z) + xT^2(y, z) \quad (1)$$

where T is the temperature, x , y , and z are the longitudinal, lateral, and transverse axes. A theory of independent cross-sectional planes of stress for prismatic beams developed by Voigt and contained in Refs. [4, 5, 6] is applied. With this theory, the following stress field is assumed for a prismatic beam with the temperature distribution given by eqn. (1)

$$x_x = y_y = z_z = y_z = x_y = x_z = 0 \quad (2)$$

The primes indicate differentiation with respect to x . The equilibrium and compatibility equations are used to obtain an expression for the normal stress. The expression for the normal stress is derived if an Airy stress function (x, y, z) linear in the x -direction is introduced as follows

$$(x, y, z) = x^1(y, z) + x^2(y, z) \quad (3a)$$

The Airy stress function defines the stresses on the cross-section as

$$y_y = \frac{\partial^2}{\partial z^2}, \quad y_z = -\frac{\partial^2}{\partial y \partial z}, \quad z_z = \frac{\partial^2}{\partial y^2} \quad (3b)$$

The materials analyzed are isotropic, homogeneous, linearly elastic, and continuous (no holes along the axis). Utilizing Boley's theory [3], where the outer surfaces of the beam are considered traction free, and following the derivation by Pilkey

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and Liu [1], the boundary conditions for are obtained as

$$\sigma_{xx} = 0 \quad \frac{\partial w_o}{\partial n} = 0 \quad \text{on } \mathbf{S} \quad (4)$$

where \mathbf{S} is the surface of the beam and \mathbf{n} is a vector perpendicular to \mathbf{S} . A relationship for and the temperature distribution is obtained if three of the thermal-stress compatibility equations are used; i.e.,

$$\sigma_{xx} = -\frac{\alpha E}{1-\nu} \Delta T \quad \text{in } \mathbf{R} \quad (5a)$$

where \mathbf{R} is the cross-sectional surface of the beam and

$$\Delta = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (5b)$$

Equation (5a) with the boundary conditions of eqn. (4) are analogous to the equations for bending of a plate with clamped edges, that is given by,

$$w_o = 0 \quad \frac{\partial w_o}{\partial n} = 0 \quad \text{on } \mathbf{S} \quad (6)$$

on the plate boundary

$$\Delta w_o = -\frac{p}{D} \quad \text{in } \mathbf{R} \quad (7)$$

in the plate interior, where w_o is the plate deflection, p is the transverse surface load on the plate and D is the plate bending rigidity. Hence, a general-purpose finite element code with plate bending elements can be used to solve the boundary value problem defined by equations eqns. (4) and (5) to determine .

The rotations of the deformed plate from the finite element code represent ,y and ,z . Numerically differentiating ,y and ,z provides ,yy and ,zz which are used in an equation for the normal stress derived by Boley [3] and Pilkey and Liu [1] from the thermal-stress compatibility equations. The normal stress is

$$\sigma_{xx} = -ET + \sigma^2 + C_1(x) + yC_2(x) + zC_3(x) \quad (8)$$

The beam is considered to be traction free at the ends such that

$$\int_A \sigma_{xx} dA = \int_A \sigma_{xx} y dA = \int_A \sigma_{xx} z dA = 0 \quad (9)$$

Thus, using eqn. (8), the functions $C_1(x)$, $C_2(x)$, and $C_3(x)$ can be defined with respect to the centroidal axis as

$$C_1(x) = \frac{P_T}{A} \quad (10a)$$

$$C_2(x) = \frac{I_y M_{Tz} - I_{yz} M_{Ty}}{I_y I_z - I_{yz}^2} \quad (10b)$$

$$C_3(x) = \frac{I_z M_{Ty} - I_{yz} M_{Tz}}{I_y I_z - I_{yz}^2} \quad (10c)$$

The coefficients $C_i(x)$, $i = 1, 2, 3$ are linear in x since the forces P_T and the moments M_{Ty} and M_{Tz} are linear in x ; that is,

$$C_i(x) = C_i^l + x C_i^2 \quad i = 1, 2, 3 \quad (10d)$$

The axial force and moments are defined by

$$P_T = \int_A (ET - \sigma^2) dA \quad (11a)$$

$$M_{Ty} = \int_A (ET - \sigma^2) z dA \quad (11b)$$

$$M_{Tz} = \int_A (ET - \sigma^2) y dA \quad (11c)$$

If one is dealing with simply-connected cross-sections, Green's theorem and the boundary conditions of eqn. (4) can be employed to reduce eqns. (11) to simple linear expressions for the end forces that are similar to eqn. (10d). However, eqns. (11) cannot be used in the elasticity formulation if the cross-section is multiply connected. Additional boundary conditions for must be imposed. The elasticity formulation just outlined will henceforth be referred to herein as the "thermoelasticity formulation."

The Airy stress function of eqn. (3a) is a linear function of x , where $\phi^1(y, z)$ could be a family of yz -intercepts for eqn. (3a) and $\phi^2(y, z)$ could be a family of slopes through the yz -planes. The biharmonic expression of eqn. (5a) can now be solved in two parts with the boundary conditions given by eqns. (4) yielding solutions for $\phi^1(y, z)$ and $\phi^2(y, z)$. A complete solution is obtained at any point along the beam by multiplying the results for $\phi^2(y, z)$ by the x -coordinate and adding $\phi^1(y, z)$.

3. Strength-of-Materials Equations

The strength-of-materials equations for the end forces and normal stress [1] are

$$P_T^* = \int_A ET dA \quad (12a)$$

$$M_{Ty}^* = \int_A (ET)z dA \quad (12b)$$

$$M_{Tz}^* = \int_A (ET)y dA \quad (12c)$$

$$\sigma_{xx} = -ET + C_1^*(x) + yC_2^*(x) + zC_3^*(x) \quad (13)$$

The eqns. (10a-d) for the thermoelasticity coefficients $C_i(x)$ are the same as the coefficients $C_i^*(x)$ for strength of materials if the end forces in eqns. (12a-c) are substituted for the end forces in eqns. (11a-c).

4. Implementation

A general-purpose finite element code was used to solve the plate bending equations analogous to eqns. (4) and (5). In particular, a plate with clamped edges and an arbitrary thickness was modeled with plate

bending elements. With this approach, the Laplacian of the temperature distribution is used as the loading. The integrals of eqns. (11) and (12) are evaluated, determining the forces and moments. The coefficients C_i and C_i^* in eqns. (10a-d) are calculated, and then used in eqns. (8) and (13) to produce the thermoelasticity and strength-of-materials normal stress, respectively.

4.1 Examples

The temperature distribution varied in the examples. Five different temperature distributions were used: cubic, exponential, logarithmic, quadratic, and linear. The equations for the temperature distribution were not calculated from the heat equation. These temperature distributions may have occurred in a short period of time or in actively cooled beam members. The shape of the cross-section varied from thin to thick with different geometric configurations.

The materials used in the examples were polycrystalline metals and a fluorocarbon rubber [7]. Polycrystalline metals are inherently isotropic [8]. The lowest Poisson's ratio considered was 0.02, while the highest was 0.499. A Poisson's ratio of 0.499 was used because a value of 0.5 generated an error in the general-purpose finite element code.

The temperature distributions in these examples were a function in only the z-direction. The simple functional dependence varied in only the z-direction, which allowed for quick verification of results. Rectangular shaped cross-sections were used because they were the simplest to verify. The origin of the yz-plane was located at the centroid of the cross-section, as shown in Figure 1.

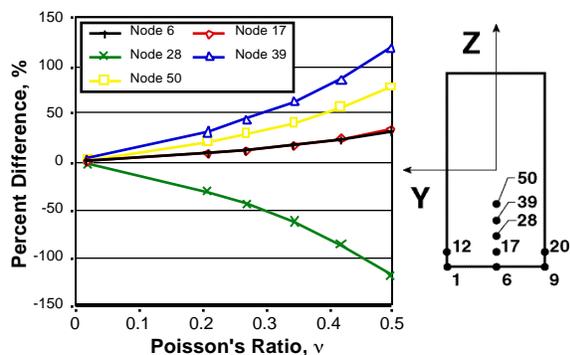


Figure 1: Percentage difference between thermoelasticity and strength-of-materials results for the cubic temperature distribution and node locations.

5. Results

The effects of temperature distribution, material properties and geometry on four of the six possible stresses were explored.

The normal stress results are compared in Table 1 by listing the percent difference between the thermoelasticity and strength-of-materials results for each temperature distribution at the various node points depicted in Figure 1. Contour plots of the normal stress and percent difference between the thermoelasticity and strength-of-materials results for the cubic temperature distribution are shown in Figure 2. The effect of Poisson's ratio is shown in Figure 1 by plotting the percentage difference between the thermoelasticity and strength-of-materials results for the cubic temperature distribution listed in Table 2. The geometric effect of the cross-sectional surface on the difference between the thermoelasticity and strength-of-materials results are listed in Table 3 for the peak normal stress and for the exponential temperature distribution.

Table 1: Percentage difference between thermoelasticity and strength-of-materials results for various temperature distributions.

Node	Exp. (%)	Cub. (%)	Log. (%)	Quad. (%)	Lin. (%)
1	0.68	0.38	0.05	0.10	0.00
6	11.6	11.7	2.91	4.17	0.00
9	6.38	5.61	1.24	1.85	0.00
12	12.4	4.65	0.57	0.57	0.00
17	13.8	12.1	2.32	2.32	0.00
20	3.34	5.89	1.57	1.57	0.00

Table 2: Percentage difference between thermoelasticity and strength-of-materials results for the cubic temperature distribution, for various Poisson's ratios.

Material	ν	Location (Node Point)				
		6 (%)	17 (%)	28 (%)	39 (%)	50 (%)
Beryllium	0.02	0.65	0.67	2.34	2.42	1.59
Chromium	0.21	8.24	8.73	31.6	31.5	20.7
Steel	0.27	11.7	12.1	44.0	43.9	28.9
Aluminum	0.345	16.7	17.3	62.6	62.5	41.1
Silicon	0.42	22.9	23.8	86.3	86.0	56.5
Rubber	0.499	31.7	32.8	119	119	78.0

6. Discussion

Various cross-sectional temperature distributions were placed upon a rectangular beam. The exponential distribution not only has a rapidly changing value, but also has a similar curve for its Laplacian. The Laplacian of the cubic temperature distribution is linear and the Laplacian of the logarithmic temperature distribution is not rapidly changing. The Laplacian of the quadratic temperature distribution is constant. The Laplacian of the linear distribution is zero; hence, the Airy stress function is zero. The exponential temperature distribution applied to the beam produces the largest stress difference between the thermoelasticity solution of this paper and the strength-of-materials solution. The cubic distribution has a small difference. The logarithmic and quadratic

distributions do not exhibit a significant difference. There is no difference between the linear distribution for the thermoelasticity solution and the strength-of-materials solution.

Poisson's ratio and ν^2 are terms not contained in the strength-of-materials formulation for the normal stress in eqn. (8) but are in eqn. (3); thus, a significant difference is expected. As Poisson's ratio increases in Table 2, the amount of error increases, producing an almost linear curve in Figure 1 when the values are plotted at different locations in the cross-section of the beam for the cubic temperature distribution, at one x-location. The remaining material properties are not a significant source of difference in results because they are contained in both the thermoelasticity and strength-of-material formulations.

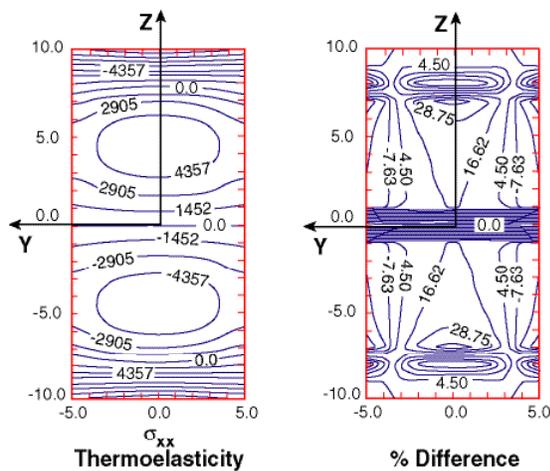


Figure 2: Contour plots for stress σ_{xx} from thermoelasticity and difference with strength-of-materials results for the cubic temperature distribution.

Table 3: Percentage difference at the peak normal stress for the exponential temperature distribution, for various geometries.

Geometry	Dimensions (y-cm by z-cm.)	Percent Diff. (%)
Thin	5 by 20	3.78
Ellipse	10 by 16	9.60
Rect. Z	10 by 20	11.64
Square	20 by 20	19.58
Rect. Y	20 by 10	19.88

As the cross-section becomes narrow in the direction perpendicular to direction of temperature variation, the percent difference decreases in Table 3. The decreasing difference between the strength-of-materials and thermoelasticity results in Table 3 can be explained by examining the method used to obtain the Airy stress function. If the surface area allowed to deflect in the plate bending problem that is used to calculate σ_{xx} is narrow, the rotations created are small, producing a small value for ν^2 . If the values of ν^2 are small in eqn. (8), the difference between the two methods becomes insignificant.

7. Conclusions

The normal-stress theory proposed by Boley has been implemented to a much higher degree than was possible in 1954 with the aid of finite elements and computers. By using the finite element method, arbitrary cross-sectional shapes are easily analyzed. The effects of temperature distribution, temperature magnitude, material properties, and geometry of the cross-section have been investigated.

The type of temperature distribution resulted in a significant difference in normal stresses obtained from the thermoelasticity and strength-of-materials formulations if the distribution has a rapidly changing T . Increasing Poisson's ratio creates greater difference in the stress for the two theories, simply because strength of materials neglects this material property. The results from thermoelasticity approach the results from strength-of-materials results as the cross-sectional shape becomes thinner.

Overall, the results in this paper show that the longitudinal normal stress can be calculated with relative efficiency with the proposed theory at any cross-sectional plane along the beam's longitudinal axis where Saint Venant's principle is valid.

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